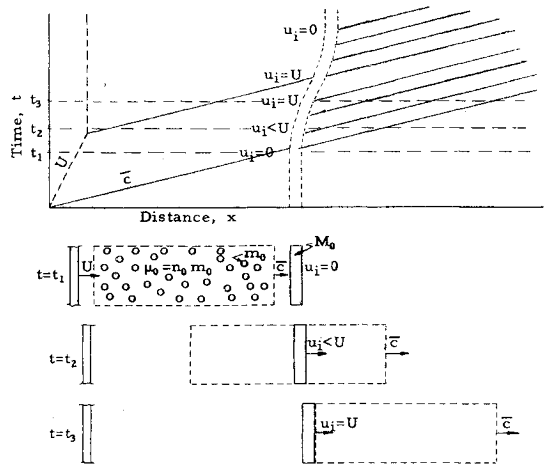

Fig. 2 Idealized collision process in rarefied gas field; $U << \bar{c}$, $\mu_0 = M_0$

tion $\psi(x)$ consequently is determined by the solution of the "wave" equation:

$$\nabla^2\psi + (8\pi^2 M_0 E_0/\beta_0^2)\psi = 0 \tag{15}$$

The free body satisfies a corresponding momentum equation of the form

$$\frac{1}{2\pi i}\frac{\partial\psi}{\partial x} \pm \frac{2\omega_0}{U}\psi = 0 \quad \text{or} \quad \frac{1}{2\pi i}\frac{\partial\psi}{\partial x} \pm \frac{M_0 U}{\beta_0}\psi = 0 \tag{16}$$

If the free body possesses a potential energy V, the total energy may be written as $E_T = \frac{1}{2}M_0U^2 + V$. Hence, $M_0U = (1/\alpha\bar{c})(E_T - V)$, and Eq. (16) can be written in the alternate form

$$\frac{\beta_0\bar{c}}{2\pi i}\frac{\partial\psi}{\partial x} \pm \frac{1}{\alpha}(E_T - V)\psi = 0 \tag{17}$$

Equation (15) is a one-dimensional classical analog of the Schrodinger equation, which also may be written in the form

$$\nabla^2\psi + (8\pi^2 M_0/\beta_0^2)(E_T - V)\psi = 0 \tag{18}$$

if the particle possesses a potential energy. Equation (17) is a classical analog of the free particle momentum equation and, if V is identified with an energy $M_0\bar{c}^2$, is a classical analog of the Dirac relativistic wave equation for a free particle.[8,9]

Further details of the characteristics of the idealized monatomic rarefied gas field described by approximate solutions of the Boltzmann equation will be described in forthcoming papers. The observations that transverse disturbances can propagate in such fields,[3,5] that the vector forms of the equations are very similar to the Maxwell equations,[6] and that classical analogs of the equations of quantum mechanics may exist suggest that the significance of these linearized rarefied gas field approximate solutions is much greater than previously has been appreciated.

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Invariant Two-Body Velocity Components and the Hodograph

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Nomenclature

e = eccentricity of conic figure
F = attracting focus of conic figure
h = specific angular momentum
r = radial distance between gravitational centers of orbital and celestial bodies
V = orbital velocity
V_h = horizontal invariant velocity component
V_p = invariant velocity component normal to the lines of apsides, $\mathbf{V} = \mathbf{V}_h + \mathbf{V}_p$
V_r = radial velocity component
V_θ = horizontal velocity component, $\mathbf{V} = \mathbf{V}_r + \mathbf{V}_\theta$
V_x = velocity component parallel to the apsidal line
V_y = velocity component normal to the apsidal line, $\mathbf{V} = \mathbf{V}_v + \mathbf{V}_y$
β = flight path angle
θ = true anomaly
μ = gravitational parameter of the two-body system

In a note published recently,[1] Cronin and Schwartz draw the readers' attention to the "little used property of two-body orbital motion, that the velocity vector at any point can be resolved into two components of invariant magnitude." They write also that "no inferences seem to have been made that there is any practical application which takes advantage of the invariance of these two velocity components."

In a series of articles presented recently,[2, 4, 5] Altman and Pistiner analyzed different two-body problems by use of the special hodograph. At the 2nd International Symposium on Rockets and Astronautics in Tokyo in 1960, Fang-Toh-Sun introduced also a special hodograph for the solution of orbital problems.[3, 6]

The purpose of this note is to show the application of the invariance of two velocity components to the derivation of both the classical (Hamilton) and the special hodograph.

Consider a two-body conic trajectory with focus F. The velocity V can be resolved (Fig. 1): 1) into two components parallel and normal to the line of apsides, $\mathbf{V} = \mathbf{V}_x + \mathbf{V}_y$; 2) into two components along and normal to the radius vector from the focus, $\mathbf{V} = \mathbf{V}_r + \mathbf{V}_\theta$; and 3) into two components of invariant magnitude V_h normal to the radiusvector and V_p normal to the line of apsides, $\mathbf{V} = \mathbf{V}_h + \mathbf{V}_p$.

Here, according to Ref. 1, $V_h = \mu/h$, a constant and everywhere normal to the radius vector, and $V_p = (\mu/h)e$, a constant and everywhere normal to the line of apsides.

From the triangle of velocities ΔABS, one has the equation of classical (Hamilton) hodograph:

$$V_x^2 + (V_y - V_p)^2 = V_h^2$$

In the plane V_x, V_y the hodograph is a circle; its center is placed on the V_y axis at a distance V_p from the origin, and its radius is equal to V_h (Fig. 2).

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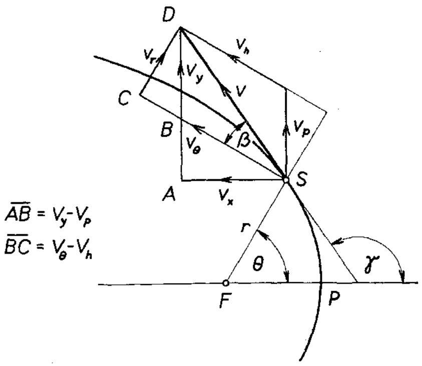

Fig. 1 Conic trajectory and velocity components

From the triangle of velocities ΔBCD, one has the equation of special hodograph:

$$(V_\theta - V_h)^2 + V_r^2 = V_p^2$$

In the plane V_θ, V_r this hodograph is a circle; its center is placed on V_θ axis at a distance V_h from the origin, and its radius is equal to V_p (Fig. 3).

It is clear that both the classical and special polar hodographs are valid for all classes of conic figures, i.e., elliptic, parabolic, and hyperbolic. Different classes of conic figures are provided by attendant changes of V_p and V_h.

For elliptical orbit,

$$e < 1 \qquad V_p < V_h$$

for parabolic orbit,

$$e = 1 \qquad V_p = V_h$$

and for hyperbolic orbit,

$$e > 1 \qquad V_p > V_h$$

Conclusions

The orbital velocity in the two-body orbital motion can be resolved into two components of invariant magnitude: V_h, normal to the radius vector, and V_p, normal to the line of apsides. These components are important for both the classical and special hodograph. In the classical hodograph, V_h is the radius of the hodograph and V_p is the ordinate of its center. In the special hodograph, V_p is the radius of the hodograph and V_h is the abscissa of its center.

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Fig. 2 Classical hodograph

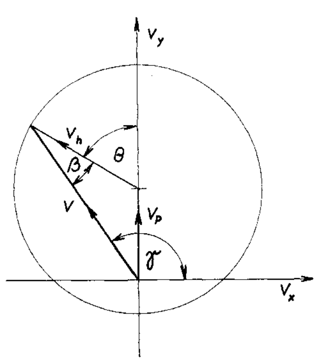
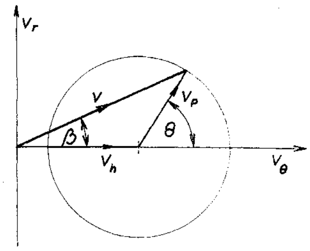


Fig. 3 Special hodograph



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Integration of the Velocity Equation of the Laminar Boundary Layer Including the Effects of Mass Transfer

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A METHOD is given for obtaining numerical solutions to the differential equation that governs fluid motion in a similar, laminar boundary layer. When some of the fluid flows in either direction through the wall, the problem contains two parameters, one being associated with the pressure gradient in the freestream while the other is a measure of the rate of mass transfer through the wall. The method of integration is quite general and gives high accuracy for any values of these parameters; it is particularly useful under conditions when other methods break down.

Spalding[1] has shown that the velocity equation of the uniform property, laminar boundary layer possesses similar solutions when the freestream velocity u_G obeys the relation

$$(du_G/dx) = C\, u_G^{2(\beta-1)/\beta} \tag{1}$$

where C and β are constants, the latter being the parameter associated with the freestream pressure gradient, and the x direction is parallel to the wall. When η the dimensionless distance from the wall, and f the dimensionless stream function are defined as

$$\eta = y\left(\frac{1}{\beta\nu}\frac{du_G}{dx}\right)^{1/2} \tag{2}$$

$$f = \frac{\psi}{u_G}\left(\frac{1}{\beta\nu}\frac{du_G}{dx}\right)^{1/2} \tag{3}$$

where y is the distance perpendicular to the wall, ν is the kinematic viscosity, and ψ the ordinary stream function, the differential equation for similar boundary layers takes the form

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{4}$$

with the boundary conditions

$$\begin{aligned} \eta &= 0 \qquad f = f_0 \qquad f' = 0 \\ \eta &\to \infty \qquad f' \to 1 \end{aligned} \tag{5}$$

The primes in Eqs. (4) and (5) denote differentiation with

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